PMM U.S.S.R.,Vol.54,No.2,pp.191-195,1990
Printed in Great Britain
0021-8928/90 $10.00+0.00

VORTEX DRAG OF A PLATE DURING VIBRATIONS IN A SLIGHTLY VISCOUS FLUID*

V.A. BUZHINSKII

The harmonic vibrations of a plate in a homogeneous incompressible fluid at rest at infinity are considered. It is assumed that the vortex flow is localized because of viscosity in a small neighbourhood of the plate edges, while the fluid stream generating this flow is described by the principal term of the expansion of the potential with a singularity for the velocity at the edge. This singularity is characterized by a velocity intensity factor. A relation is set up between this factor and the kinetic energy of the fluid and methods to determine it are examined. Necessary conditions for the existence of the hypothesized fluid flow are clarified. Asymptotic dependences are obtained for the energy of vortex formation and the drag coefficient when these conditions are satisfied. A comparison is made with experimental data.

1. The velocity intensity factor. We consider an ideal incompressible fluid flow without vortices past an extended infinitely thin plate with an arbitrary smooth surface and boundary contour (Fig.1). We select a point on the boundary at which we draw a tangent plane to the plate surface. We let $\mathbf{n}$ and $\boldsymbol{\tau}$ denote unit vectors along the normal and tangent to the boundary contour in this plane. We define the unit vector of the binormal $\mathbf{e} = \boldsymbol{\tau} \times \mathbf{n}$. We connect a local rectangular coordinate system Oxy whose x and y axes are directed along $\mathbf{n}$ and $\mathbf{e}$, respectively, to each point of the boundary.

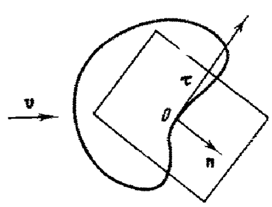

Fig.1

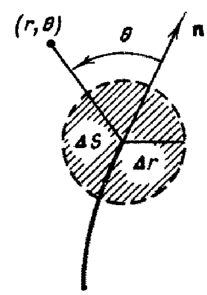

Fig.2

The fluid flow in a small neighbourhood of the plate edge is almost planar, hence the complex potential and velocity in the xy plane can be represented in the form /1/

$$w = Az^{1/2},\ \bar{v} = {}^1/_2 Az^{-1/2},\ z = x + iy \tag{1.1}$$

where A is a complex constant for the selected point of the contour. If the fluid flow is non-stationary, then A depends on the time t, which plays the role of a parameter. The fluid velocity has a singularity at the point $z = 0$. This singularity is characterized by a power and the quantity A. The velocity along $\boldsymbol{\tau}$ is finite at any point.

In addition to the complex coefficients A we introduce real coefficients according to the formula

$$K_v^2 = {}^1/_2 \pi A \bar{A} \tag{1.2}$$

and we denote by K_v the velocity intensity factor (VIF). We will show that the square of the VIF characterizes the fluid kinetic energy density T near the plate edge. Using the asymptotic representation (1.1) and the definition (1.2), we obtain

$$\Delta T/\Delta l = {}^1/_2 \rho K_v^2 \Delta r$$

where ρ is the fluid density, Δl is a small length element for the edge, and Δr is the

*Prikl.Matem.Mekhan.,54,2,233-238,1990

radius of a small circle with centre at the point O (Fig.2).

2. ***Determination of the VIF.*** The coefficients A and K_v depend on the location of the point on the boundary contour, the plate geometry, and the domain occupied by the fluid, as well as on the boundary conditions.

We will examine the plane problem of the flow past a fixed plate of infinite span by an ideal incompressible fluid whose velocity $\mathbf{v}_\infty$ at infinity is constant in magnitude and direction. The problem can be inverted by considering that the plate moves while the fluid is at rest at infinity. If the function

$$z = f(\zeta) = {}^1/_2\zeta + c_0 + c_1\zeta^{-1} + \ldots \tag{2.1}$$

realizes the conformal mapping of the exterior of the plate in the physical complex z plane into the exterior of a circle of radius R in an auxiliary complex plane ζ, then /2/

$$dw/dz = {}^1/_2\,(\bar{\mathbf{v}}_\infty - \mathbf{v}_\infty R^2\zeta^{-2})d\zeta/dz \tag{2.2}$$

It can be assumed that the plate edge is at a point z_* on the real axis while the point $\zeta_* = Re^{-i\alpha}$ is its image in the plane ζ where α is a certain angle. In a small neighbourhood of the edge

$$d\zeta/dz = g\,(z_*)(z - z_*)^{-1/2} \tag{2.3}$$

Substituting (2.3) into (2.2) and comparing with (1.1) for $\zeta = \zeta_*$, we obtain

$$A = -2ig\,(z_*)v_\infty \sin(\theta + \alpha)e^{i\alpha} \tag{2.4}$$

where θ is the angle between the direction of the velocity and the real axis.

The coefficient A can be evaluated from (2.4) when the mapping (2.1) is known. Thus, for a straight plate

$$K_v{}^2 = {}^1/_2\pi R v_\infty{}^2 \sin^2\theta \tag{2.5}$$

where R is the plate width, and θ is the angle between the plate and the velocity. For the arc of a circle of radius R with half the aperture angle 2β

$$K_v{}^2 = \pi R v_\infty{}^2 \sin^2(\theta + \beta)\sin 2\beta \tag{2.6}$$

where θ is the angle between the chord of the arc and the velocity. It follows from (2.4) that a direction $\theta = -\alpha$, always exists for which $A = 0$ and $K_v = 0$.

We will obtain a relation that enables the VIF calculation to be simplified in a number of cases. Let us decrease the size of the plate by displacing points of the boundary contour l by the same amount δn opposite to $\mathbf{n}$ while keeping the pressure unchanged on different sides of the surface of the discontinuity formed. Let us change to local coordinates connected to the new edge while retaining the previous notation x and y for them. Every potential motion of a homogeneous incompressible fluid can be considered to originate from a state of rest because of an impact /3/; consequently, the change in fluid kinetic energy after removal of the pressure will be

$$\delta T = {}^1/_2 i \oint_l \int_0^{\delta n} p_t v\, dx\, dl, \quad p_t = -\rho\,(w_+ - w_-)$$

where p_t is the pressure pulse. The values of the velocity v and the potentials w_+ and w_- on different sides of the surface are determined from (1.1), where the potential is determined by replacing z by $z + \delta n$.

Evaluating the integral over x we find

$$\delta T/\delta n = {}^1/_2\rho \oint K_v{}^2 dl \tag{2.7}$$

Application of the dependence obtained is especially effective in cases when the form of the change in K_v is known along the length of the contour l from symmetry considerations. It is possible to calculate K_v by changing part of the plate boundary contour and evaluating the integral in (2.7) on only this part of the contour.

As an example we examine the application of the dependence (2.7) to determine K_v for motion perpendicular to its plane by a thin circular disc in an unbounded fluid at rest at infinity. The fluid kinetic energy will be

$$T = {}^1/_2\mu v_0{}^2 = {}^4/_3\rho R^3 v_0{}^2 \tag{2.8}$$

where μ is the apparent mass, and v_0 and R are the disc velocity and radius. Assuming $\delta n = \delta R$ and substituting (2.8) into (2.7) we find

$$K_v{}^2 = 4\pi^{-1}R v_0{}^2 \tag{2.9}$$

In the general case, numerical methods must be relied upon to determine the VIF. It is also useful here to utilize the dependence (2.7) to refine the results obtained. An analytical solution can be obtained for an elliptic plate, which is of interest in that K_v depends on the location of a point on the boundary contour. For flat plates K_v obviously depends only on the velocity component normal to the plate; consequently, it is sufficient to consider the plate motion perpendicular to its plane. The solution of this problem is analogous to the determination of the stress intensity factors for an elliptic crack in a solid body, known from linear fracture mechanics (/4/, pp.149-156), consequently

$$K_v^2 = \frac{\pi v_0^2}{E^2(k)} \frac{b}{a} (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)^{1/2}, \quad k^2 = 1 - \frac{b^2}{a^2} \tag{2.10}$$

where a and b are the semimajor and semiminor axes of the ellipse, φ is an angle governing the parametric coordinates of a point on the ellipse: $x = a \cos \varphi$, $y = b \sin \varphi$; $E(k)$ is the complete elliptic integral of the second kind. At the ends of the semimajor axis K_v^2 is b/a times less than at the ends of the semiminor axis. Relations (2.5) and (2.9) are obtained as special cases from (2.10) for $k = 1$ and $k = 0$.

3. ***Energy of vortex formation during plate vibrations in a fluid.*** Consider the harmonic oscillations of a plate in an unbounded fluid at rest at infinity. The radii of curvature of the surface and boundary contour of the plate are considered to be quantities of the same or greater order than its characteristic transverse dimension. We will write the complex potential of the fluid flow in the neighbourhood of the edge in the form

$$w = Az^{1/2} \cos \omega t \tag{3.1}$$

where ω is the vibration frequency. As above, only the principal term of the expansion that results in a singularity for the velocity and consequently plays a special role is retained in the complex potential. Unlike (1.1), the time dependence is extracted explicitly here; consequently, the amplitude values are denoted by A and K_v.

We will now assume that the flow past the plate occurs so that the quite complex vortex flow occurring at all points of its contour is localized because of viscosity in a small neighbourhood of the edge; outside this neighbourhood the representation (3.1) holds asymptotically. Let us clarify the condition necessary for this.

The governing parameters characterizing fluid flow in the neighbourhood of the edge are: the fluid density ρ, the vibration frequency ω, the fluid kinematic viscosity ν and the coefficient K_v. In addition to the characteristic dimension R of the plate we introduce the linear scale $\delta = (\nu/\omega)^{1/2}$, that characterizes the thickness of the oscillating boundary layer, and the linear scale, $d = (K_v/\omega)^{2/3}$, that characterizes the dimension of the vorticity domain near the sharp edge.

Satisfaction of the conditions

$$\delta/R \ll d/R \ll 1 \tag{3.2}$$

is necessary for the assumed flow to exist.

If conditions (3.2) are not satisfied, then the vortex flow cannot be considered as an imposition on a flow with the complex potential (3.1).

We note that $K_v \sim R^{1/2} v_0$, where v_0 is the characteristic amplitude of the plate velocity, and consequently $d/R \sim (v_0/\omega R)^{2/3}$. It hence follows that the condition $d/R \ll 1$ is satisfied for large Strouhal numbers $\mathrm{Sh} = \omega R/v_0$ or, equivalently, for small relative amplitudes of the plate vibrations. Conditions (3.2) can be written more roughly (in terms of the global flow characteristics) in the form

$$\mathrm{Re}^{-1/2} \ll \mathrm{Sh}^{-2/3} \ll 1, \quad \mathrm{Re} = \omega R^2/\nu$$

If conditions (3.2) are satisfied, then the VIF K_v is a unit parameter characterizing the fluid flow in the "far" neighbourhood of the edge and the formation of the vorticity domain in the "near" neighbourhood of the edge. Consequently, the dimensions of a very small vorticity domain depend asymptotically on K_v and are explicitly independent of the geometric parameters of the plate and the boundary conditions far from the edge. It is clear that the assumption made here that the fluid occupies a boundless space is not essential. Converging vorticity reconstructs the flow near the edge in such a way that the velocity singularity caused by the main inviscid flow with complex potential (3.1) is eliminated. Vorticity domains are generally not formed on the sections of the plate boundary where $K_v = 0$ for a separation-free potential flow.

Assuming conditions (3.1) to be satisfied, we will determine the energy dE of vortex formation in a small element dl of the plate boundary contour in the vibration period. A single dimensionless combination

$$\pi_1 = K_v^{4/3} / (\nu \omega^{-1/3}) \tag{3.3}$$

equivalent to the Reynolds number of the local flow, can be constructed from the governing parameters ρ, ω, ν and K_v. Applying the π-theorem of similarity /5/, we obtain

$$dE / dl = B(\pi_1)\, \rho K_v^{4/3} \omega^{-2/3} \tag{3.4}$$

Since (3.3) reduces to the form $\pi_1 = d^2/\delta^2$ it follows from (3.2) that $\pi_1 \gg 1$. Consequently, it can be assumed that the coefficient B does not depend on the Reynolds number, i.e., it is a constant.

Experimental investigations /6, 7/ show that the vortex drag during plate vibrations in a fluid is independent, in practice, of the Reynolds number as it varies over a broad range of values $Re = 10^3 - 10^6$. The total energy of vortex formation during the period of vibration is found by integrating (3.4) over the boundary contour

$$E = B\rho\omega^{-2/3} \int K_v^{4/3} dl \tag{3.5}$$

The mode of plate motion was not made specific when deriving the dependence (3.5) and can be arbitrary, as long as the assumptions made are not disturbed. The elastic vibrations of a plate are an important special case.

4. Vortex resistance during harmonic oscillations of a plate. Available experimental data /7/ can be represented in the form of the dependence of the average (on the basis of equality of work during the vibration period) of the drag coefficient c_x on the Strouhal number $\omega R/v_0$, since no noticeable dependence of c_x on the Reynolds number has been found. We will represent the drag force acting on the plate in the form /6/

$$F = -{}^1/_2 c_x \rho S\, |\mathbf{v}|\, \mathbf{v}, \quad \mathbf{v} = \mathbf{v}_0 \cos \omega t$$

where S is the area of the middle section. Equating the work of this force per period of vibrations to the energy of vortex formation (3.5) we find

$$c_x = K\,(R\omega/v_0)^{1/3} \tag{4.1}$$

$$K = \frac{3}{4} B \frac{R^2}{S} \oint \left(\frac{K_v^2}{Rv_0^2} \right)^{2/3} d\,\frac{l}{R} \tag{4.2}$$

Formula (4.1) yields the dependence of the plate drag resistance coefficient on the relative amplitude of the vibrations. This dependence is quite different from that in /8/, obtained for small plate vibrations in a fluid flow with a significant velocity.

The VIF K_v can be determined theoretically. Consequently, the single unknown quantity in (4.2) remains the coefficient B. Since this coefficient is a universal constant under the assumptions made, it can be found if the dependence of c_x on the vibration amplitude is determined numerically or experimentally for any one plate.

Values of c_x were determined experimentally for rings and long plates of constant width under forced vibrations in a direction perpendicular to their plane. These results were approximated by the dependence $c_x = K\,[v_0/(R\omega)]^n$, and the values of K and n were found by least squares. The dependence (4.1) with $K \approx 4.6$ was obtained, in particular, for conditions corresponding to plate vibrations in an unbounded fluid.

Determining K_v by (2.5) for $\theta = {}^1/_2\pi$ in the case under consideration and substituting it into (4.2), we obtain $K = {}^3/_2\,({}^1/_2\pi)^{2/3} B$. Hence $B \approx 1.7$.

Consequently, if conditions (3.2) are satisfied approximately, the problem of determining the energy of vortex function and the vortex drag during harmonic vibrations of a plate reduces to determining the VIF within the framework of the concept of separation-free potential fluid flow.

Let us consider two examples. We will compare the vortex drag for vibrations in a direction normal to its plane for a free plate of infinite span and for the same plate arranged by one of its infinite edges in an infinite plane wall. Putting $K_v^2 = \pi R v_0^2$ into (4.2), we obtain $K = 5.7$. Comparing with $K = 4.6$ for a free plate, we find that the plate drag at the wall is 1.25 times greater, which is in agreement with experimental data /7/.

We will determine the vortex drag of a thin circular disc performing vibrations perpendicular to its plane. Substituting (2.9) into (4.2), we find $K = 3.5$.

We present a comparison with experimental data obtained by I.M. Mel'nikova in tests on a circular plate of diameter 1.5m in an air medium. The logarithmic damping decrement of the free plate vibrations on an elastic suspension was determined in these tests. For weak damping, the decrement δ of the vibrations can be calculated as the ratio between the vortex formation energy (3.5) and twice the total vibration energy, which yields

$$\delta = K\left(\frac{v_0}{\omega R}\right)^{2/3}, \quad K = B\,\frac{\rho R^3}{\mu} \oint \left(\frac{K_v^2}{Rv_0^2}\right)^{2/3} d\,\frac{l}{R} \tag{4.3}$$

where μ is the generalized mass taking the suspension and the apparent mass of the air into account.

The experimental data represented in Fig.3 are obtained for the frequency $\omega/(2\pi) = 0.4 s^{-1}$ and corresponds to $\mu = 11.3$ kg. The dependence (4.3) computed for this case by using (2.9) is displayed in Fig.3 by a curve that lies below the test data, where the discrepancy grows as the vibrations amplitude increases, and reaches 25% for $v_0/(\omega R) = 0.16$.

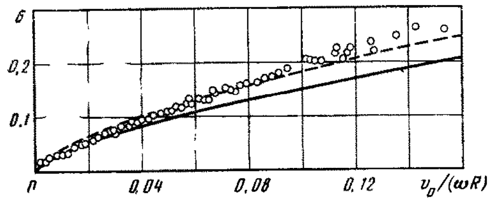

Fig.3

If the asymptotic nature of the dependence (4.3) is taken into account, the agreement with experimental data can be considered satisfactory. The discrepancy noted above can here be explained by the following reasoning. The error in determining the universal constant B by the experimental data in /7/ is around 15% because of their spread. The test data represented in Fig.3 are more accurate but a certain influence of small stiffener ribs that were fastened to the plate to eliminate its elastic vibration was not taken into account in the computation. Taking the above into account, it is more correct to take $B = 2$, roughly. The dependence corresponding to this value of B is shown dashed in Fig.3.

The author is grateful to particular at the seminar supervised by G.G. Chernyi for discussing the research and for useful remarks.

REFERENCES

1. BATCHELOR G., Introduction to the Dynamics of Fluids, Mir, Moscow, 1973.
2. LOITSYANSKII L.G., Mechanics of Fluids and Gases. Nauka, Moscow, 1970.
3. SEDOV L.I., Mechanics of a Continuous Medium, 2, Nauka, Moscow, 1973.
4. LIEBOWITZ H. Ed., Fracture, 2, Mir, Moscow, 1975.
5. SEDOV L.I., Similarity and Dimensional Analysis Methods in Mechanics, Nauka, Moscow, 1987.
6. KEULEGAN G.H. and CARPENTER L.H., Forces on cylinders and plates in an oscillating fluid, J. Res. Nat. Bur. Standards, 60, 5, 1958.
7. MIKISHEV G.N. and RABINOVICH B.I., Dynamics of Thin-walled Structures with Compartments Containing Fluids. Mashinostroyeniye, Moscow, 1971.
8. SEDOV L.I., Plane Problems of Hydrodynamics and Aerodynamics. Nauka, Moscow, 1966.

Translated by M.D.F.